SOME ESTIMATES RELATED TO OH'S CONJECTURE FOR THE CLIFFORD TORI IN $\mathbb{C}P^n$

EDWARD GOLDSTEIN

Abstract. This note is motivated by Y.G. Oh's conjecture that the Clifford torus L_n in $\mathbb{C}P^n$ minimizes volume in its Hamiltonian deformation class. We show that there exist explicit positive constants a_n depending on the dimension with $a_2 = 3/\pi$ such that for any Lagrangian torus L in the Hamiltonian class of L_n we have $vol(L) \geq a_n vol(L_n)$. The proof uses the recent work of C.H. Cho [Cho] on Floer homology of the Clifford tori. A formula from integral geometry enables us to derive the estimate. We wish to point out that a general lower bound on the volume of L exists from the work of C. Viterbo [Vit]. Our lower bound $a_2 = 3/\pi$ is the best one we know.

1. Introduction

The Clifford torus L_n in $\mathbb{C}P^n$ is given in homogeneous coordinates by

$$\big((z_1 : \ldots : z_{n+1})\big| |z_i| = |z_j|\big)$$

The Clifford torus is the only orbit of the diagonal torus action on $\mathbb{C}P^n$ which is a minimal Lagrangian submanifold, see [Gold1]. It is also the only orbit which is a monotone Lagrangian submanifold, see [CG]. Y.G. Oh has studied the second variation of volume of L_n with respect to Hamiltonian deformations, see [Oh]. He has shown that this variation is non-negative and conjectured that L_n minimizes volume in its Hamiltonian deformation class. This note constitutes an effort toward verifying this conjecture. Our main tool is the recent result of Cheol-Hyun Cho [Cho] which states that if L is Hamiltonian equivalent to L_n and if L and L_n intersect transversally then the number of intersection points of L and L_n

$$\#(L \bigcap L_n) \geq 2^n$$

We will use integral geometry to study the volume of such L - see also [IOS] for a similar usage of integral geometry for a product of two geodesics in $S^2 \times S^2$. Our main result is that

$$vol(L) \geq a_n vol(L_n)$$

with an explicit positive constant a_n and $a_2 = \frac{3}{\pi}$.

2. A formula from integral geometry

The presentation here follows R. Howard [How]. In our case the group $SU(n+1)$ acts on $\mathbb{C}P^n$ with a stabilizer $K \simeq U(n)$. Thus we view $\mathbb{C}P^n = SU(n+1)/K$ and the Fubini-Study metric is induced from the bi-invariant metric on $SU(n+1)$. Let P and Q be two Lagrangian submanifolds of $\mathbb{C}P^n$. For a point $p \in P$ and $q \in Q$ we define an angle $\sigma(p,q)$ between the tangent plane T_pP and T_qQ as follows: First we choose some elements g and h in $SU(n+1)$ which move p and q respectively to the same point $r \in \mathbb{C}P^n$. Now the tangent planes g_*T_pP and h_*T_qQ are in the

same tangent space $T_r\mathbb{C}P^n$ and we can define an angle between them as follows: take an orthonormal basis $u_1 \dots u_n$ for g_*T_pP and an orthonormal basis $v_1 \dots v_n$ for h_*T_qQ and define

$$\sigma(g_*T_pP, h_*T_qQ) = |u_1 \wedge \dots \wedge v_n|$$

The later quantity $\sigma(g_*T_pP, h_*T_qQ)$ depends on the choices g and h we made. To mend this will need to average this out by the stabilizer group K of the point r. Thus we define:

$$\sigma(p,q) = \int_K \sigma(g_*T_pP, k_*h_*T_qQ)dk$$

Since $SU(n+1)$ acts transitively on the Grassmanian of Lagrangian planes in $\mathbb{C}P^n$ we conclude that this angle is a constant depending just on n:

$$\sigma(p,q) = c_n$$

There is a following general formula due to R. Howard [How]:

$$\int_{SU(n+1)} \#(gP \bigcap Q)dg = \int_{P\times Q} \sigma(p,q)dpdq = c_n vol(P)vol(Q)$$

Thus

$$vol(P)vol(Q) = \frac{1}{c_n}\int_{SU(n+1)} \#(gP \bigcap Q)dg \tag{1}$$

The quantity of interest for us is the constant $\frac{vol(SU(n+1))}{c_n}$. We'll find it using $P = Q = \mathbb{R}P^n$.

3. The case of $\mathbb{R}P^n$

Let P be $\mathbb{R}P^n$ and let Q be Hamiltonian equivalent to P. It is known that if P and Q intersect transversally then $\#(P \bigcap Q) \geq n+1$- see [Giv] and also [FOOO] for a more general treatment of fixed point sets of antisymplectic involutions. On the other hand if g is a unitary matrix then linear algebra shows that $\#(gP \bigcap P) = n+1$ (again assuming transversality). Thus there is a proposition due to B. Kleiner:

Proposition 1. *(Kleiner) $\mathbb{R}P^n$ minimizes volume in its Hamiltonian isotopy class*

For our purposes we are interested in plugging the formula 1 with $P = Q = \mathbb{R}P^n$. We conclude that

$$\frac{vol(SU(n+1))}{c_n} = \frac{vol(\mathbb{R}P^n)^2}{n+1} \tag{2}$$

Let us work out the case $n = 2$. The metric on $\mathbb{C}P^2$ is the quotient of the metric on S^5 by S^1-action. We have $vol(\mathbb{R}P^2) = vol(S^2)/2 = 2\pi$. So

$$\frac{vol(SU(3))}{c_2} = 4\pi^2/3$$

4. The estimate for the Clifford torus

Let $L_n \subset \mathbb{C}P^n$ be the Clifford torus. There is a torus $T^{n+1} \subset \mathbb{C}^{n+1}$ given by

$$T^{n+1} = \big((z_1, \ldots, z_{n+1})\big| |z_i| = 1/\sqrt{n+1}\big)$$

We have that L_n is the quotient of T^{n+1} by the S^1 action. Thus

$$vol(L_n) = vol(T^{n+1})/2\pi = (2\pi/\sqrt{n+1})^{n+1}/2\pi$$

For $n = 2$ we have

$$vol(L_2) = 4\pi^2/3\sqrt{3}$$

Let P be Hamiltonian equivalent to L_n. From [Cho] we have that for a unitary matrix g: $\#(gP \bigcap P) \geq 2^n$. Thus from equations 1 and 2 we conclude that

$$vol(P)^2 \geq 2^n \frac{vol(SU(n+1))}{c_n} = 2^n \frac{vol(\mathbb{R}P^n)^2}{n+1}$$

Let us specialize to the case $n = 2$. We have

$$vol(P)^2 \geq 4 \cdot 4\pi^2/3 = (4\pi)^2/3$$

Thus

$$vol(P) \geq 4\pi/\sqrt{3} = \frac{3}{\pi} vol(L_2)$$

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E-mail: egold@math.stanford.edu